Painlevé tests, singularity structure and integrability

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Summary. After a brief introduction to the Painlevé property for ordinary differential equations, we present a concise review of the various methods of singularity analysis which are commonly referred to as Painlevé tests. The tests are applied to several different examples, and the connection between singularity structure and integrability for ordinary and partial differential equation is discussed.

1 Introduction

The connection between the integrability of differential equations and the singularity structure of their solutions was first discovered in the pioneering work of Kowalevski [65], who considered the equations for the motion under gravity of a rigid body about a fixed point, namely

$$\begin{aligned} \frac{d\underline{\ell}}{dt} &= \underline{\ell}\times\underline{\omega} + \underline{c}\times\underline{g}\,, \\ \frac{d\underline{g}}{dt} &= \underline{g}\times\underline{\omega}\,; \qquad\qquad \underline{\ell} = \mathbf{I}\,\underline{\omega}\,. \end{aligned} \tag{1}$$

In the above, $\underline{\ell}$ and $\underline{\omega}$ are respectively the angular momentum and angular velocity of the body, $\underline{g}$ is the gravity vector with respect to a moving frame, and the centre of mass vector $\underline{c}$ and inertia tensor $\mathbf{I}$ are both constant. The remarkable insight of Kowalevski was that the system of equations (1) could be solved explicitly whenever the dependent variables $\underline{\ell}$ and $\underline{g}$ are *meromorphic* functions of time t extended to the complex plane, $t \in \mathbb{C}$. By requiring that the solutions should admit Laurent expansions around singular points, she found constraints on the constants $\underline{c}$ and $\mathbf{I} = \mathrm{diag}(I_1, I_2, I_3)$ (diagonalized in a suitable frame). Her method isolated the two solvable cases previously known to Euler ($\underline{c} = 0$) and Lagrange ($I_1 = I_2$ with $\underline{c}$ defining the axis of symmetry), as well as a new case having principal moments of inertia $I_1 = I_2 = 2I_3$ and $\underline{c}$ perpendicular to the axis of symmetry. The latter case is now known as the Kowalevski top, and Kowalevski was further able to integrate it explicitly

in terms of theta-functions associated with a hyperelliptic curve of genus 2, thereby proving directly that the solutions are meromorphic functions of t. A modern discussion can be found in [8] or [69], for instance.

An important feature of the equations (1) from the point of view of singularity analysis is that they are *nonlinear*. For a linear differential equation

$$\frac{d^n y}{dz^n} + a_{n-1}(z)\frac{d^{n-1}y}{dz^{n-1}} + \ldots + a_1(z)\frac{dy}{dz} + a_0(z)y = 0,$$

of arbitrary order n it is well known [43, 55] that the general solution can have only *fixed* singularities at the points in the complex z-plane where the coefficient functions $a_j(z)$ are singular. However, for nonlinear differential equations, as well as the fixed singularities which are determined by the equation itself, the solutions can have *movable* singularities which vary with the initial conditions. For example, the first order nonlinear differential equation

$$\frac{dy}{dz} + y^2 = 0$$

has the general solution

$$y = \frac{1}{z - z_0}, \qquad z_0 \quad \text{arbitrary},$$

with a movable simple pole at $z = z_0$. If the initial data $y = y_0$ is specified at the point $z = 0$, then the position of the simple pole varies according to

$$z_0 = -\frac{1}{y_0}.$$

The classification of ordinary differential equations (ODEs) in terms of their singularity structure was initiated in the work of Painlevé [78]. The main property that Painlevé sought for ODEs was that their solutions should be single-valued around movable singular points. Nowadays this property is usually formulated thus:

Definition 1.1. The Painlevé property for ODEs: *An ODE has the Painlevé property if all movable singularities of all solutions are poles.*

Painlevé proved that for first order ODEs of the general form

$$y' = \frac{\mathcal{P}(y,z)}{\mathcal{Q}(y,z)},$$

where $\mathcal{P}$ and $\mathcal{Q}$ are polynomial functions of y and analytic functions of z (and the prime $'$ denotes d/dz), then the only movable singularities that can arise are poles and algebraic branch points. The latter are excluded by Definition 1.1, and he further showed that the most general first order equation with the Painlevé property is the Riccati equation

$$y' = a_2(z)y^2 + a_1(z)y + a_0(z),$$

where the coefficients a_0, a_1, a_2 are analytic functions of z.

For second order ODEs, life is more complicated because movable essential singularities can arise (see e.g. chapter 3 in [6] for an example). Painlevé initiated the classification of second order ODEs of the form

$$y'' = \mathcal{F}(y', y, z). \tag{2}$$

with $\mathcal{F}$ being a rational function of y' and y, and analytic in z. Painlevé and his contemporaries succeeded in classifying all ODEs of the type (2) which fulfill the requirements of Definition 1.1. The complete result is in the form of a list of approximately fifty representative equations, unique up to Möbius transformations, which are detailed in chapter 14 of Ince's book [55]. It was found that (after suitable changes of variables) all of these ODEs have general solutions in terms of classical special functions (defined by linear equations) or elliptic functions, except for six special equations which are now known as Painlevé I-VI (or just PI-VI).

As an example, consider the second order ODE

$$y'' = 6y^2 - \frac{1}{2}g_2. \tag{3}$$

This can be immediately integrated once, because the equation is autonomous (the right hand side is independent of z), which yields

$$(y')^2 = 4y^3 - g_2 y - g_3, \tag{4}$$

with g_3 being an integration constant. The general solution of the first order ODE (4) is given by the Weierstrass elliptic function,

$$y = \wp(z - z_0; g_2, g_3) \tag{5}$$

with the constants g_2, g_3 being the invariants. The solution (5) has infinitely many movable double poles, at $z = z_0$ and at all congruent points $z = z_0 + 2m\omega_1 + 2n\omega_2 \in \mathbb{C}$ for $(m, n) \in \mathbb{Z}^2$ on the period lattice defined by the half-periods ω_1, ω_2. (For an introduction to Weierstrass elliptic functions see chapter 20 in [102] or chapter VI in [91].) We shall return to the equation (3) in the next section.

The first of the *Painlevé equations* is PI, which is a non-autonomous version of (3) given by

$$y'' = 6y^2 + z. \tag{6}$$

Its general solution is a meromorphic function of z, and the solution of PI (or sometimes the equation itself) may be referred to as a Painlevé transcendent, since it essentially defines a new transcendental function. The other equations PII-PVI also contain parameters; for example the second Painlevé equation (PII) is

$$y'' = 2y^3 + zy + \alpha \tag{7}$$

where α is a constant parameter. The general solution of each of the Painlevé equations cannot be expressed in terms of elliptic functions or other classical special functions [57], although for special parameter values they can be solved in this way; e.g. when α is an integer, equation (7) has particular solutions given by rational functions of z, and it has solutions in terms of Airy functions for half-integer values of α.

An important early result was the connection of PVI with the isomonodromic deformation of an associated linear system [31]. After the work of Painlevé and his colleagues around the turn of the last century, the Painlevé equations were probably only of interest to experts on differential equations. However, in the latter half of the 20th century the Painlevé transcendents enjoyed something of a renaissance when it was discovered that they gave exact formulae for correlation functions in solvable models of statistical mechanics [103], quantum field theory [56] and random matrix models [27, 59], and also arose as symmetry reductions of partial differential equations (PDEs) integrable by the inverse scattering transform (see [4] and section 3 below). The link with integrable PDEs and linear Lax pairs established the exact solution of the Painlevé equations by the isomonodromy method [30]. More recently a weakened version of the Painlevé property has been used to find exact metrics for relativistic fluids [39]. With this wide variety of physical applications, the Painlevé transcendents have acquired the status of nonlinear special functions (see the review and references in chapter 7 of [5]).

The continuation of Painlevé's classification programme to higher order equations becomes increasingly difficult as the order increases. Even at third order a new phenomenon can be encountered, in the form of a movable natural barrier or boundary beyond which the solution cannot be analytically continued; this occurs in Chazy's equation

$$y''' = 2yy'' - 3(y')^2. \tag{8}$$

A variety of results for third or higher order equations have been obtained by Chazy [14], Gambier, Bureau, and most recently by Cosgrove; see [23] and references therein. Chazy's equation (8) has some higher order relatives known as Darboux-Halphen systems, which have a very complicated singularity structure, and occur as reductions of the integrable self-dual Yang-Mills equations (see the contribution of Ablowitz *et al* in [22]).

It should be clear from the above that the Painlevé property has a very deep connection with the concept of integrability. This connection is by no means straighforward, and continues to be the subject of active research [22]. In the rest of this brief review article, we will introduce the basic techniques for testing the singularity structure of differential equations (both ODEs and PDEs), which are often referred to collectively as *Painlevé analysis*. The basic method for testing ODEs by expansions in Laurent series is treated in section 2. This method should probably be referred to as the *Kowalevski-Painlevé test* to honour both pioneers of the subject, but most commonly

only Painlevé is mentioned in this context. Section 3 describes the conjecture of Ablowitz, Ramani and Segur [4] on the connection between integrable PDEs and Painlevé-type ODEs, and how this can be used as an integrability test for PDEs. In the fourth section we explain how the preceding analysis can be bypassed by a direct consideration of the singularity structure of a PDE, by using the method of Weiss, Tabor and Carnevale [100]. This is followed in section 5 by associated truncation techniques related to Bäcklund transformations, Lax pairs and Hirota bilinear equations for integrable systems of PDEs. In section 6 we highlight the limitations of the Painlevé property as a criterion for integrability, in the context of integrable systems with movable algebraic branching and the weak Painlevé property [86]. In the final section we give our outlook on methods of singularity analysis for differential equations, and mention how some of these methods and concepts have been extended to the discrete context of maps or difference equations.

2 Painlevé analysis for ODEs

Consider an ODE for a dependent variable $y(z)$, which may be a single scalar, or a vector quantity. If the ODE has the Painlevé property then its solutions must have local Laurent expansions around movable singularities at $z = z_0$, where z_0 is arbitrary. However, if branching occurs then this can be detected by local singularity analysis. The basic Painlevé test for ODEs consists of the following steps:

- **Step 1:** Identify all possible *dominant balances*, i.e. all singularities of form $y \sim c_0\,(z - z_0)^{\mu}$.
- **Step 2:** If all exponents μ are integers, find the *resonances* where arbitrary constants can appear.
- **Step 3:** If all resonances are integers, check the *resonance conditions* in each Laurent expansion.
- **Conclusion:** If no obstruction is found in steps 1-3 for every dominant balance then the Painlevé test is satisfied.

Note that the exponents μ and leading coefficients c_0 must have as many components as the vector y, and if the ODE is polynomial then at least one of the exponents must be a *negative* integer for a leading order pole-type singularity. Rather than give formal definitions of the terms introduced in steps 1-3 above (which can be found in [13] and elsewhere in the references), we would like to illustrate them with a couple of examples.

First of all we describe the Painlevé test applied to the equation (3), in which case y is just a scalar. Applying step 1 we look for leading order behaviour which produces a singularity in the ODE, so we require $y \sim c_0\,(z-z_0)^{\mu}$ and μ must be a negative integer for a movable pole with no branching. This gives immediately

$$y \sim \frac{1}{(z-z_0)^2} \tag{9}$$

as the only possible dominant balance. Note that we could have also obtained this balance by assuming that y blows up as $z \to z_0$, and then (since g_2 is constant) $y^2 >> g_2$ on the right hand side of the ODE, so the y^2 term must balance with the left hand side of (3), giving

$$y'' \sim 6y^2, \qquad \text{as} \quad z \to z_0. \tag{10}$$

We can multiply by y' on both sides of (10) and integrate to find

$$\frac{1}{2}(y')^2 \sim 2y^3, \qquad \text{as} \quad z \to z_0 \tag{11}$$

(throwing away the integration constant, which is strictly dominated by the other terms), and after taking a square root in (11) and integrating we find (9).

We now seek a solution of (3) given locally by a Laurent expansion around a double pole at $z = z_0$, in the form

$$y = \sum_{j=0}^{\infty} c_j \, (z-z_0)^{j-2}, \qquad c_0 = 1, \tag{12}$$

where the value of c_0 has been fixed as in (9). We wish to determine the *resonances*, which are the positions in the Laurent series (12) where arbitrary coefficients c_j can appear. Since the ODE (3) is second order, there must be two arbitrary constants in a local representation of the general solution: z_0, the arbitrary position of the movable pole, and one other. To apply step 2 of the Painlevé test we take a perturbation of the leading order with small parameter ϵ, in the form

$$y \sim (z-z_0)^{-2}(1+\epsilon(z-z_0)^r). \tag{13}$$

To first order in ϵ we have

$$y^2 \sim (z-z_0)^{-4}(1+2\epsilon(z-z_0)^r), \qquad y'' \sim (z-z_0)^{-4}(6+\epsilon(r-2)(r-3)(z-z_0)^r).$$

Thus when we substitute the perturbation (13) into the dominant terms (10) and retain only first order terms in ϵ we find

$$y'' - 6y^2 \sim \epsilon\Big((r-2)(r-3) - 12\Big)(z-z_0)^{r-4} = 0.$$

Since the perturbation ϵ is arbitrary, corresponding to the first appearance of a new arbitrary constant in the Laurent expansion (12), the expression in large brackets must vanish, giving the resonance polynomial

$$r^2 - 5r - 6 = 0, \qquad \text{whence} \quad r = -1 \quad \text{or} \quad r = 6.$$

The first resonance at $r = -1$ must *always* be present in any expansion around a movable singularity, since it corresponds to the arbitrariness of z_0. The second resonance value $r = 6$ indicates that the coefficient c_6 should be arbitrary.

In order to complete the Painlevé test, we must now substitute in the full Laurent expansion and check that it is consistent up to the coefficient c_6. In this case we find that the expansion is precisely

$$y = \frac{1}{(z-z_0)^2} + \frac{1}{20}g_2(z-z_0)^2 + \frac{1}{28}g_3(z-z_0)^4 + \ldots, \tag{14}$$

so that $c_6 = g_3/28$ is the arbitrary constant that appears after integrating (3) to obtain (4). In fact only even powers of $(z-z_0)$ occur in this expansion, since the Weierstrass function (5) is an even function of its argument. The higher coefficients in (14) can be found recursively in terms of the invariants g_2, g_3. (Up to overall multiples these coefficients are the Eisenstein series associated to the corresponding elliptic curve [91].) The pole position z_0 does not appear in the coefficients because the ODE (3) is autonomous.

Here we should point out that passing the basic Painlevé test is only a *necessary* condition for an ODE to have the Painlevé property. Proving the Painlevé property requires showing that the local Laurent expansions can be analytically continued globally to a single-valued function (or one with only fixed branched points), in the absence of movable essential singularities. For the ODE (3) this follows from the fact that the general solution (5) is given by a Weierstrass elliptic function, which is meromorphic (for a proof see e.g. [91, 102]). Painlevé's proof that the first Painlevé transcendent (6) is free from movable essential singularities is outlined by Ince in chapter 14 of [55], but the proof is unclear and this has prompted recent efforts to find a more straightforward approach [44, 61, 93].

Having seen an example where the Painlevé test is passed, we now move on to an example for which it fails, by considering the following coupled second order system:

$$y_1'' = 2y_1^2 - 12y_2, \qquad y_2'' = 2y_1y_2. \tag{15}$$

In [35] this system is associated to an interaction of four particles moving in a plane, subject to velocity-dependent forces, and in that context it is essential that both $y_1(z)$, $y_2(z)$ (denoted $c_2(\tau)$, $c_4(\tau)$ in the original reference) and the independent variable z should be *complex*. To find the dominant balances, we look for leading order singular behaviour of the form

$$y_1 \sim aZ^\mu, \qquad y_2 \sim bZ^\nu, \tag{16}$$

corresponding to a singularity in the solution at $Z = z - z_0 = 0$ for at least one of μ, ν negative. Because the system (15) is *autonomous*, we can expand in the variable Z, since the position z_0 of the movable singularity will not appear in the coefficients of local expansions around $z = z_0$.

There are three possible dominant balances for the system (15), namely

$$\begin{array}{rlll} (i) & y_1 \sim 3Z^{-2}, & y_2 \sim bZ^{-2}, & b \ \text{arbitrary;} \\ (ii) & y_1 \sim 3Z^{-2}, & y_2 \sim bZ^{3}, & b \ \text{arbitrary;} \\ (iii) & y_1 \sim 10Z^{-2}, & y_2 \sim \frac{35}{3}Z^{-4}. & \end{array}$$

Other possible power law behaviour around $Z = 0$ corresponds to μ, ν both non-negative integers and leads to Taylor series expansions, which are not relevant to our analysis of singular points.

The second step in applying the Painlevé test is to find the resonances. For the system (15) to possess the Painlevé property we require that all resonances for all dominant balances must be integers, and at least one balance must have one resonance value of -1 with the rest being non-negative integers, in which case this is a *principal balance* for which the Laurent expansion should provide a local representation of the general solution. To find the resonance numbers r we substitute

$$y_1 \sim aZ^{\mu}(1 + \delta Z^r), \qquad y_2 \sim bZ^{\nu}(1 + \epsilon Z^r)$$

into the dominant terms of the system (15) for each of the balances $(i) - (iii)$, and take only the terms linear in δ and ϵ. This yields a pair of homogeneous linear equations for δ, ϵ (which correspond to the arbitrary coefficients appearing at the resonances). The determinant of this 2×2 system must vanish, which gives in each case a fourth order polynomial in r.

Principal balance (i): It turns out that the balance (i) is the only principal balance, with resonances

$$(i) \qquad r = -1, 0, 5, 6.$$

As mentioned before, the resonance -1 is always present, since it corresponds to the arbitrary position z_0 of the pole, while $r = 0$ comes from the arbitrary constant b in the leading order term of the expansion for y_2; the other two values arise from arbitrary coefficients higher up in the series for y_1, y_2, so that altogether there should be four arbitrary constants appearing in these Laurent series. However, for step 3 of the test we also require that all resonance conditions hold: so far we have only found the orders in the series where arbitrary constants may appear, but it is necessary to check that all other terms vanish at this order when the series are substituted into the equations. Taking

$$y_1 \sim L_1(Z) := \sum_{j=-2}^{\infty} k_{1,j} Z^j, \qquad y_2 \sim L_2(Z) := \sum_{j=-2}^{\infty} k_{2,j} Z^j \tag{17}$$

in the each of the equations (15) we know already that the leading order terms require

$$k_{1,-2} = 3, \qquad k_{2,-2} = b \ (\text{arbitrary}),$$

giving the resonant term at $r = 0$ in the expansion for y_2, while at subsequent orders we find

$$k_{1,-1} = 0 = k_{2,-1}; \quad k_{1,0} = b,\, k_{2,0} = -b^2/3; \quad k_{1,1} = 0 = k_{2,1}.$$

At the next orders we further obtain

$$k_{1,2} = -3b^2/5,\, k_{2,2} = 7b^3/15; \quad k_{1,3} = 0,\, k_{2,3} \quad \text{arbitrary},$$

so that the resonance condition at $r = 5$ corresponding to $k_{2,3}$ is satisfied. However, at the next order in the first equation of the system (15), at the first appearance of the resonance coefficient $k_{1,4}$, we find the additional relation

$$k_{2,2} = -b^3/5,$$

which means that the resonance condition is not satisfied unless $b = 0$, contradicting the fact that b should be arbitrary. Thus the Painlevé test is failed by this principal balance.

The only way to rectify the failure of the resonance condition and leave b as a free parameter is to modify (17) by adding logarithm terms. More precisely taking

$$y_1 \sim L_1(Z) + \Delta_1(Z), \qquad y_2 \sim L_2(Z) + \Delta_2(Z), \tag{18}$$

the resonance condition is resolved by taking

$$\Delta_1 \sim -\frac{8}{7} b^3 Z^4 \log Z, \qquad \Delta_2 \sim -\frac{8}{21} b^4 Z^4 \log Z. \tag{19}$$

However, the additional terms Δ_1, Δ_2 in (18) must then consist of a doubly infinite series in powers of Z and $\log Z$, with the leading order behaviours given by (19). Only in this way is it possible to represent the general solution of the system (15) as an expansion in the neighbourhood of a singular point containing four arbitrary parameters. Such infinite logarithmic branching is a strong indicator of non-integrability [73, 87].

Non-principal balance (ii): The second balance denoted (ii) above has resonances

$$r = -5, -1, 0, 6.$$

The presence of the negative integer value $r = -5$ means that this is a non-principal balance. (For an extensive discussion of negative resonances see [20].) This gives Laurent expansions

$$y_1 \sim 3Z^{-2} + kZ^4 - \frac{3}{2} bZ^5 + O(Z^7), \qquad y_2 \sim bZ^3 + O(Z^5). \tag{20}$$

In this case all resonance conditions are satisfied and all higher coefficients in (20) are determined uniquely in terms of k and b. However, because it only contains three arbitrary constants (namely b, k and the position z_0 of the pole), it cannot represent the general solution, but can correspond to a particular solution which is meromorphic.

Non-principal balance (iii): For the balance (iii) the resonances are given by $r = -1$ and the roots of the cubic equation

$$r^3 - 15r^2 + 26r + 280 = 0,$$

which turn out to be a real irrational number and a complex conjugate pair, approximately

$$r = -3.2676, \quad 9.1338 \pm 1.5048i.$$

While non-integer rational resonances are allowed within the weak extension of the Painlevé test (see [86] and section 6), irrational or complex resonances lead to infinite branching, and (as already evidenced by the principal balance (i)) the system (15) cannot possess the Painlevé property. This non-principal balance may be interpreted as a particular solution corresponding to a degenerate limit of the general solution, and perturbation of this particular solution (within the framework of the Conte-Fordy-Pickering perturbative Painlevé test [20]) will pick up the logarithmic branching present in the general solution. Clearly it would have been sufficient to stop the test after the failure of the resonance condition in the principal balance (i), but we wanted to present the details of the other balances to show the different possibilities that can arise in the singularity analysis of ODEs.

3 The Ablowitz-Ramani-Segur conjecture

Having considered how to test for the Painlevé property in ODEs, we now turn to the connection with integrable PDEs. In the 1970s it was discovered that ODEs of Painlevé type, and in particular some of the Painlevé transcendents, appeared as symmetry reductions of PDEs solvable by the inverse scattering technique. This led Ablowitz, Ramani and Segur [4] to formulate the following.

Ablowitz-Ramani-Segur conjecture: *Every exact reduction of a PDE which is integrable (in the sense of being solvable by the inverse scattering transform) yields an ODE with the Painlevé property, possibly after a change of variables.*

To obtain ODE reductions of PDEs one can use the classical Lie symmetry method or its non-classical variants (see [77] for details), or the direct method of Clarkson and Kruskal [16]. The idea is that having found the symmetry reductions of the PDE, one can either solve the ODEs that are obtained, or apply the Painlevé test to them, to see if branching occurs. If all the ODE reductions are of Painlevé type, then this suggests that the original PDE may be integrable. However, the need to allow for a possible change of variables will become apparent in section 6. Indeed, the most difficult aspect of this conjecture, if one would like to provide a proof of it, is in defining exactly what class of variable transformations should be allowed.

As an example, consider the Korteweg–de Vries (KdV) equation for long waves on shallow water, which we write in the form

$$u_t = u_{xxx} + 6uu_x. \tag{21}$$

This has three essentially different reductions to ODEs; details of their derivation are given in chapter 3 of [77]. The first is the travelling wave solution

$$u(x,t) = w(z), \qquad z = x - ct, \tag{22}$$

where c is the (arbitrary) wave speed and $w(z)$ satisfies

$$w''' + 6ww' + cw' = 0. \tag{23}$$

After an integration and a shift in w this is equivalent to (3), and the solution of (23) is given by

$$w = -2\wp(z - z_0) - c/6, \tag{24}$$

where z_0 and the invariants g_2 and g_3 of the $\wp$-function are arbitrary constants. In the special case $g_2 = 4k^4/3$, $g_3 = -8k^6/27$ the elliptic function degenerates to a hyperbolic function, and for $c = -4k^2$ the reduction (22) yields the one-soliton solution

$$u(x,t) = 2k^2 \mathrm{sech}^2(kx + 4k^3 t). \tag{25}$$

(Of course there is the additional freedom to shift the position of the soliton (25) by the transformation $x \to x - x_0$.)

The second reduction of KdV is the Galilean-invariant solution

$$u(x,t) = -2\left(w(z) + t\right), \qquad z = x - 6t^2 \tag{26}$$

where $w(z)$ satisfies

$$w''' - 12ww' - 1 = 0. \tag{27}$$

Upon integration, and making a shift in z to remove the constant of integration, the ODE (27) becomes the first Painlevé equation (6).

The third reduction of (21) is the scaling similarity solution

$$u(x,t) = (-3t)^{-\frac{2}{3}} w(z), \qquad z = (-3t)^{-\frac{1}{3}} x. \tag{28}$$

This solution arises from the invariance of the PDE (21) under the group of scaling symmetries

$$(x,t,u) \longrightarrow (\lambda x, \lambda^3 t, \lambda^{-2} u).$$

After substituting the similarity form (28) into KdV and integrating once we find the ODE for w:

$$w'' + 2w^2 - zw + \frac{\ell^2 - 1/4 + w' - (w')^2}{2w - z} = 0. \tag{29}$$

The parameter ℓ^2 is the constant of integration, and (29) turns out to be equivalent to the equation P34, so called because it is labelled $XXXIV$ in the Painlevé classification of second order ODEs as detailed by Ince [55]. The equation P34 can be solved in terms of the second Painlevé equation (7), according to the relation

$$w = -y' - y^2, \qquad \text{with} \quad \ell = \alpha + 1/2. \tag{30}$$

The above formula defines a Bäcklund transformation between the two equations (29) and (7), and in fact there is a one-one correspondence between their solutions. With the parameters of the two ODEs related as in (30), the inverse of this transformation (defined for $w \neq z/2$) is given by

$$y = \frac{w' + \alpha}{2w - z}.$$

For more details, and higher order analogues, see [49] and references.

Thus we have seen that the ODE reductions of the KdV equation (21) are solved either in elliptic functions or in terms of Painlevé transcendents, and hence these reductions certainly have the Painlevé property. So the KdV equation clearly fulfills the necessary condition for integrability required by the Ablowitz-Ramani-Segur conjecture, as it should do because it is integrable by means of the inverse scattering transform. In contrast to KdV, we consider another equation that models long waves in shallow water, namely the Benjamin-Bona-Mahoney (often referred to as BBM) equation [9], which takes the form

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{31}$$

The Benjamin-Bona-Mahoney equation is also known as the regularized long-wave equation, and was apparently first proposed by Peregrine [79]. The travelling wave reduction of the Benjamin-Bona-Mahoney equation is very similar to that for KdV: the PDE (31) has the solution

$$u(x,t) = -12c\wp(z - z_0) + c - 1, \qquad z = x - ct, \tag{32}$$

given in terms of the Weierstrass $\wp$-function (with arbitrary values of the invariants g_2, g_3 and the constant z_0). In the hyperbolic limit with $c = (1 - 4k^2)^{-1}$ for $k \neq \pm 1/2$ this gives the solitary wave solution

$$u(x,t) = \frac{12k^2}{1 - 4k^2} \operatorname{sech}^2(kx - k(1 - 4k^2)^{-1}t),$$

but in contrast to (25) this is not a soliton because the Benjamin-Bona-Mahoney equation is not integrable and collisions between such waves are inelastic: see the discussion and references in chapter 10 of [26].

Evidence for the non-integrable nature of the Benjamin-Bona-Mahoney equation is provided by another symmetry reduction, namely

$$u(x,t) = \frac{1}{t} w(z) - 1, \qquad z = x + \kappa \log t, \tag{33}$$

where κ is a constant. Upon substitution of (33) into (31), w is found to satisfy the ODE

$$\kappa w''' - w'' - ww' - \kappa w' + w = 0. \tag{34}$$

For all values of the parameter κ, the equation (34) does not have the Painlevé property, which means that (at least in these variables) the Benjamin-Bona-Mahoney equation fails the necessary condition required by the Ablowitz-Ramani-Segur conjecture. In the case $\kappa = 0$, (34) just becomes second order, so it is possible to compare with the list in chapter 14 of Ince's book [55] to see that $w'' + ww' - w = 0$ is not an ODE of Painlevé type. A direct method, which works for any κ, is to apply Painlevé analysis directly to the equation and show that a resonance condition is failed. In fact the analysis can be greatly simplified by integrating in (34) to obtain

$$\kappa w'' - w' - \frac{w^2}{2} - \kappa w = -\int_{z_1}^{z} w(s)\, ds. \qquad (35)$$

(The lower endpoint of integration z_1 is an arbitrary constant.)

Now we can perform a Painlevé test on the integro-differential equation (35). For $\kappa \neq 0$, in the neighbourhood of a movable singularity at $z = z_0$ the dominant balance is between the w'' and w^2 terms, giving

$$w(z) \sim 12\kappa(z - z_0)^{-2}, \qquad z \to z_0.$$

If we suppose that this is the leading order in a Laurent expansion around $z = z_0$, i.e.

$$w(z) = \sum_{j=0}^{\infty} w_j (z - z_0)^{j-2}, \qquad w_0 = 12\kappa, \qquad (36)$$

then at the next order we see that the coefficient of $(z - z_0)^{-1}$ is

$$w_1 = \frac{12\kappa}{6\kappa - 1}, \qquad \kappa \neq 1/6.$$

(For $\kappa = 1/6$ the Laurent expansion immediately breaks down.) However, substituting the expansion (36) into the left hand side of (35) gives a Laurent series, while on the right hand side there is a term $\log(z - z_0)$ arising from the nonzero residue $w_1 \neq 0$. Hence the expansion (36) cannot satisfy the equation (35), or equivalently (34), and the Painlevé test is failed.

Thus we have seen that all of the ODE reductions of the KdV equation possess the Painlevé property, but not all the reductions of the non-integrable Benjamin-Bona-Mahoney equation (31) are of Painlevé type. We leave it as an exercise for the reader to check whether the Benjamin-Bona-Mahoney equation has other reductions apart from (32) and (33) (for hints see exercise 3.2 in [77]). However, it should be clear from the above that a fair amount of work is required when analysing a PDE in the light of the Ablowitz-Ramani-Segur conjecture, since one must first find all possible reductions to ODEs and then perform Painlevé analysis on each of them separately. Finding the symmetry reductions can be a difficult enterprise in itself (see [17] for example), but in the next section we shall see how this complication can be avoided by using the direct method due to Weiss, Tabor and Carnevale [100].

4 The Weiss-Tabor-Carnevale Painlevé test

While the symmetry reductions of PDEs are clearly indicative of their integrability or otherwise, it is more convenient to analyse the singularity structure of PDEs directly. This approach was pioneered by Weiss, Tabor and Carnevale [100] (hence it is usually referred to as the WTC Painlevé test). However, in the context of PDEs with d independent (complex) variables $z_1, \ldots, z_d$ the singularities of the solution no longer occur at isolated points but rather on an analytic hypersurface $\mathcal{S}$ of codimension one, defined by an equation

$$\phi(\mathbf{z}) = 0, \qquad \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d, \tag{37}$$

where ϕ is analytic in the neighbourhood of $\mathcal{S}$. The hypersurface where the singularities lie is known as the *singular manifold*, and it can be used to define a natural extension of the Painlevé property for PDEs, which we state here in the form given by Ward [98]:

Definition 4.1. The Painlevé property for PDEs: *If $\mathcal{S}$ is an analytic non-characteristic complex hypersurface in $\mathbb{C}^d$, then every solution of the PDE which is analytic on $\mathbb{C}^d \backslash \mathcal{S}$ is meromorphic on $\mathbb{C}^d$.*

With the above definition in mind, it is natural to look for the solutions of the PDE in the form of a Laurent-type expansion near $\phi(\mathbf{z}) = 0$:

$$u(\mathbf{z}) = \frac{1}{\phi(\mathbf{z})^\mu} \sum_{j=0}^{\infty} \alpha_j(\mathbf{z})\, \phi(\mathbf{z})^j, \tag{38}$$

If the PDE has the Painlevé property, then the leading order exponent μ appearing in the denominator of (38) should be a positive integer, with the expansion coefficients α_j being analytic near the singular manifold $\phi = 0$, and sufficiently many of these must be arbitrary functions together with the arbitrary non-characteristic function ϕ. As mentioned in [60] in the context of the self-dual Yang-Mills equations, and further explained in [98], it is important to state that ϕ should be non-characteristic because (even for linear equations) the solutions of PDEs can have arbitrary singularities along characteristics.

The application of the Weiss-Tabor-Carnevale test using series of the form (38) proceeds as for the usual Painlevé test for ODEs: when the series is substituted into the PDE, equations arise at each order in ϕ which determine the coefficients α_j succesively, except at resonant values $j = r$, where the corresponding α_r are required to be arbitrary (subject to compatibility conditions being satisfied). The Weiss-Tabor-Carnevale test is only passed if all resonance conditions are fulfilled for every possible balance in the PDE (i.e. all consistent choices of μ). Note that, just as for ODEs, passing the test merely constitutes a *necessary* condition for the Painlevé property: a complete proof is much harder in general, although in the particular case of the self-dual Yang-Mills equations Ward [98] was able to use twistor methods to prove that they satisfy the requirements of Definition 4.1.

To see how the Weiss-Tabor-Carnevale test works, we will indicate the first steps of the analysis for the example of the KdV equation (21). In that case, there are just two independent variables x and t, so $d = 2$, and there is only one dominant balance where the degree of the singularity for the linear term u_{xxx} matches that for the nonlinear term uu_x. Substituting an expansion of the form (38) into (21), with $\mathbf{z} = (x, t) \in \mathbb{C}^2$, it is clear that this gives $\mu = 2$ as the only possibility, and for the leading order and next-to-leading order the first two coefficients are determined as

$$\alpha_0 = -2\phi_x^2, \qquad \alpha_1 = 2\phi_{xx}. \tag{39}$$

This means that the expansion around the singular manifold for KdV can be written concisely as

$$u(x,t) = 2(\log\phi)_{xx} + \sum_{k=0}^{\infty} \alpha_{k+2}(x,t)\,\phi(x,t)^k, \tag{40}$$

where it is necessary to assume $\phi_x \not\equiv 0$ so that ϕ is non-characteristic.

In general, at each order j there is a determining equation for the coefficients of the series given by

$$(j+1)(j-4)(j-6)\alpha_j = F_j[\phi_x, \phi_t, \phi_{xt}\ldots, \alpha_k;\, k < j], \tag{41}$$

where the functions F_j depend only on the previous coefficients α_k for $k < j$ and their derivatives, as well as the various x and t derivatives of ϕ. It is clear from (41) that the resonance values are $r = -1, 4, 6$, meaning that we require ϕ, α_4 and α_6 to be arbitrary functions of x and t. For the KdV equation, apart from the standard resonance at -1 corresponding to the arbitrariness of ϕ, the other necessary conditions for $r = 4, 6$, namely $F_4 \equiv 0$, $F_6 \equiv 0$ are satisfied identically, and so in accordance with the Cauchy-Kowalevski theorem these three arbitrary functions are the correct number to provide a local representation (40) for the general solution of the third order PDE (21). We leave it to the reader to calculate the expressions for the higher F_j in (41) and verify the resonance conditions for F_4 and F_6; this is a standard calculation, so we omit further details which can be found in several sources, e.g. [73, 100]. For completeness we note that the issue of convergence of the expansion (40) for KdV has also been completely resolved [62].

We shall return briefly to the KdV equation in the next section, where we discuss how series such as (38) can be truncated within the *singular manifold method*, leading to Bäcklund transformations and Lax pairs for integrable PDEs, and by further truncation to Hirota bilinear equations for the associated tau-functions. Before doing so, we would like to illustrate ways in which the basic Weiss-Tabor-Carnevale test may be further simplified, taking the non-integrable Benjamin-Bona-Mahoney equation (31) as our example. Applying the test as outlined above directly to the equation (31) leads to an expansion (38) very similar to that for KdV: it also has a single dominant balance with $\mu = 2$ for a non-characteristic singular manifold (where

$\phi_x \not\equiv 0 \not\equiv \phi_t$), and the same resonances $r = -1, 4, 6$, but for the Benjamin-Bona-Mahoney equation not all resonance conditions are satisfied and the test is failed. It is a good exercise to perform this calculation and compare it with the corresponding results for KdV. Rather than presenting such a comparison here, we wish to give two shortcuts to the conclusion that the equation (31) does not possess the Painlevé property for PDEs. First of all, observe that if $\phi_x \not\equiv 0$ then locally we can apply the implicit function theorem and solve the equation (37) for x. Thus we set

$$\phi(x,t) = x - f(t) \tag{42}$$

with $\dot{f}(t) := df/dt \not\equiv 0$, and then we can take the coefficients in the expansion (38) to be functions of t only; this is referred to as the 'reduced ansatz' of Kruskal, first suggested in [60]. With this ansatz, the Weiss-Tabor-Carnevale analysis for PDEs becomes only slightly more involved than applying the Painlevé test for ODEs, and so constitutes a very effective way to decide if a PDE is likely to be integrable.

For the Benjamin-Bona-Mahoney equation there is a second shortcut that can be made, which is to take the potential form of the equation by making use of the fact that it has a conservation law. This approach is widely applicable, since nearly all physically meaningful PDEs admit one or more conservation laws. For the equation (31) it is immediately apparent that it can be put in conservation form as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u_{xt} - \frac{1}{2}u^2 - u\right),$$

which implies that

$$C = \int_{-\infty}^{\infty} u\,dx$$

is a conserved quantity for the Benjamin-Bona-Mahoney equation, i.e. $dC/dt = 0$ for $u(x,t)$ defined on the whole real x-axis with vanishing boundary conditions at $x = \pm\infty$. It follows that upon introducing the potential v as the new dependent variable, with

$$v = \int_{-\infty}^{x} u\,dx \longrightarrow C \quad \text{as} \quad x \to \infty,$$

we can replace u by v and its derivatives in (31) to obtain the potential form of the PDE, namely

$$v_t - v_{xxt} + v_x + \frac{1}{2}v_x^2 = 0 \tag{43}$$

(where we have integrated once and applied the boundary conditions to eliminate the arbitrary function of t). If we now apply the Weiss-Tabor-Carnevale test to (43), at the same time using the 'reduced ansatz' (42), then we see that the only possible leading exponent in a Laurent-type expansion for v is $\mu = 1$, giving

$$v(x,t) = \sum_{j=0}^{\infty} \beta_j(t)\,(x - f(t))^{j-1}. \tag{44}$$

The equations for the coefficients $\beta_j(t)$ at each order take the form

$$(j+1)(j-1)(j-6)\beta_j = F_j[\dot{f}, \ddot{f}, \ldots, \beta_k;\ k < j],$$

so the resonances are $r = -1, 1, 6$ which compares with $r = -1, 4, 6$ for the original equation (31): clearly one of the resonances has shifted to a lower value by taking the equation in potential form (43). Upon substituting the series (44) into the potential Benjamin-Bona-Mahoney equation, the leading order term is at order ϕ^{-4}, giving the equation

$$-6\beta_0 \dot{f} + \frac{1}{2}\beta_0^2 = 0.$$

Since $\beta_0 \not\equiv 0$, this determines the first coefficient as

$$\beta_0 = 12\dot{f}.$$

However, at the next order ϕ^{-3} in the equation (43), we have the resonance $r = 1$ with the condition

$$-2\dot{\beta}_0 = 0, \qquad \text{whence} \qquad \ddot{f} = 0. \tag{45}$$

Since f is supposed to be an arbitrary non-constant function of t, we see that the resonance condition (45) is not satisfied, so the equation (43) fails the Weiss-Tabor-Carnevale Painlevé test, indicating the non-integrability of the Benjamin-Bona-Mahoney equation. However, observe what happens if f is a linear function of t: then (45) is satisfied, corresponding to the travelling wave reduction (32), which does have the Painlevé property.

The only way to remove the restriction (45) on the function f would be to add a term $-(\dot{\beta}/\dot{f})\log(x - f(t))$ to the expansion (44). It has been observed [83] that the inclusion of terms linear in $\log\phi$ for PDEs in potential form is not incompatible with integrability. However, in this case terms of all powers of $\log(x - f(t))$ are required to ensure a consistent expansion in the potential Benjamin-Bona-Mahoney equation (43) with three arbitrary functions f, β_1 and β_6 corresponding to the three resonances.

For the reader who is interested in applying either the Painlevé test for ODEs, as described in section 2, or the Weiss-Tabor-Carnevale Painlevé test for PDEs, it is worth remarking that software implementations of these tests are now freely available. The web page `www.mines.edu/fs_home/whereman` has algorithms written by D. Baldwin and W. Hereman, for instance.

5 Truncation techniques

Aside from the obvious application of the various Painlevé tests in isolating potentially integrable equations (for example, in the classification of integrable

coupled KdV equations [63]), their usefulness can be extended by the means of truncation techniques. The first of these is known as the *singular manifold method*, which was primarily developed in a series of papers by Weiss [101]. The idea behind the method is that by truncating an expansion such as (38), usually at the zero order (ϕ^0) term, it is possible to obtain a Bäcklund transformation for the PDE. For such truncated expansions the singular manifold function ϕ is no longer arbitrary, but satisfies constraints. In the case of integrable equations that are solvable by the inverse scattering transform, the singular manifold method can be used to derive the associated Lax pair; for directly linearizable equations, such as Burger's equation or its hierarchy [80], the method instead leads to the correct linearization. Even for non-integrable PDEs, where the constraints on ϕ are much stronger, the singular manifold method can still be used to obtain exact solutions. Furthermore, for integrable PDEs the truncation approach can be carried further by cutting off the series *before* the zero order term, to yield tau-functions satisfying bilinear equations [33].

We will outline the basic truncation results for the KdV equation (21), before presenting more detailed calculations for the nonlinear Schrödinger (NLS) equation. For KdV, the Laurent-type expansion (40) can be consistently truncated at the zero order term to yield

$$u = 2(\log\phi)_{xx} + \tilde{u}, \qquad \tilde{u} \equiv \alpha_2. \tag{46}$$

While substituting the full expansion (40) into KdV gives an infinite set of equations (41) for ϕ and the α_j, the truncated expansion gives only a finite number. The last of these equations does not involve ϕ, and just says that $\tilde{u}$ is also a solution of KdV, i.e.

$$\tilde{u}_t = \tilde{u}_{xxx} + 6\tilde{u}\tilde{u}_x.$$

The other equations (after some manipulation and integration) boil down to just two independent equations for ϕ and $\tilde{u}$, as follows:

$$\tilde{u} = k^2 - \frac{(\sqrt{\phi_x})_{xx}}{\sqrt{\phi_x}}; \tag{47}$$

$$\frac{\phi_t}{\phi_x} = 6k^2 + \left(\frac{\phi_{xxx}}{\phi_x} - \frac{3\phi_{xx}^2}{2\phi_x^2}\right). \tag{48}$$

In the above, k is a constant parameter. The important feature to note is that since u and $\tilde{u}$ are both solutions of (21), the equation (46) constitutes a Bäcklund transformation for KdV, provided that ϕ satisfies (47) and (48). For example, starting from the seed solution $\tilde{u} = 0$, the Bäcklund transformation defined by (46), (47) and (48) can be used to generate the one-soliton solution (25), or even a mixed rational-solitonic solution by taking $\phi = (x - 12k^2t) + (2k)^{-1}\sinh(2kx + 8k^3t)$.

It is maybe not immediately obvious that the system comprised of the two equations (47) and (48) is equivalent to the standard Lax pair for KdV. This can be seen by making the squared eigenfunction substitution $\phi_x = \psi^2$, so that (47) becomes a linear (time-independent) Schrödinger equation. In the context of quantum mechanics in one dimension, ψ is the wave function with potential $-\tilde{u}$ and energy $-k^2$, i.e. (47) is equivalent to

$$\psi_{xx} + \tilde{u}\psi = k^2\psi.$$

The second equation (48) is known as the Schwarzian KdV equation [74], and in its own right it constitutes a nonlinear integrable PDE for the dependent variable ϕ; with the squared eigenfunction substitution it leads to the linear equation for the time evolution ψ_t. All these results for KdV are well known, and have been extended to the whole KdV hierarchy; the interested reader who wishes to check these calculations is referred to [73] for more details.

Perhaps less well understood, however, is the interesting connection [33] between the singularity structure of PDEs and the tau-function approach to soliton equations pioneered by Hirota [46, 75], which culminated in the Sato theory relating integrable systems to representations of affine Lie algebras [71, 76]. The link with the singular manifold method is made by truncating the expansion (40) at the last singular term in ϕ, and setting $\phi = \tau$, to give

$$u = 2(\log\tau)_{xx}, \tag{49}$$

which is the standard substitution for the KdV variable u in terms of its tau-function. From (21), after substituting (49) and performing an integration (subject to suitable boundary conditions), a bilinear equation is obtained for the new dependent variable τ. This bilinear equation may be written concisely as

$$(D_xD_t - D_x^4)\tau\cdot\tau = 0, \tag{50}$$

by making use of the Hirota derivatives:

$$D_x^jD_t^k g\cdot f := \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^j \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k g(x,t)f(x',t')|_{x'=x,t'=t}.$$

The bilinear form is particularly convenient for calculating multi-soliton solutions [46], and leads to the connection with vertex operators [71, 75, 76]. For solitons the tau-function is just a polynomial in exponentials. In general τ is holomorphic, so from (49) it is clear that the places where τ vanishes correspond to the singularities of u.

We now present details on the application of the singular manifold method to the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0. \tag{51}$$

This PDE (commonly referred to as NLS) describes the evolution of a complex wave amplitude ψ, and due to the minus sign in front of the cubic nonlinear

term this is the non-focusing case of the nonlinear Schrödinger equation; the focusing case has $+2|\psi|^2\psi$ instead, and describes a different physical context. The following results on the singular manifold method for the nonlinear Schrödinger equation appeared in [47]. Seeking an expansion of the form (38) for (51), at leading order we find the behaviour

$$\psi \sim \frac{\alpha_0}{\phi}, \qquad |\alpha_0|^2 = \phi_x^2.$$

Thus, truncating the expansion at the zero order (ϕ^0) level, we find

$$\psi = \frac{\alpha_0}{\phi} + \hat{\psi}, \qquad \hat{\psi} \equiv \alpha_1. \tag{52}$$

To proceed with the singular manifold method we substitute the truncated expansion (52) into (51), and set the terms at each order in ϕ to zero. This yields the following four equations (the singular manifold equations):

$$\begin{aligned}
\phi^{-3} :& \qquad |\alpha_0|^2 - \phi_x^2 = 0; \\
\phi^{-2} :& \; i\phi_t + 2\phi_x(\log\alpha_0)_x + \phi_{xx} + 2\alpha_0\overline{\hat{\psi}} + 4\overline{\alpha}_0\hat{\psi} = 0; \\
\phi^{-1} :& \qquad i\alpha_{0,t} + \alpha_{0,xx} - 4\alpha_0|\hat{\psi}|^2 - 2\overline{\alpha}_0\hat{\psi}^2 = 0; \\
\phi^{0} :& \qquad i\hat{\psi}_t + \hat{\psi}_{xx} - 2|\hat{\psi}|^2\hat{\psi} = 0.
\end{aligned} \tag{53}$$

Clearly the coefficient of ϕ^{-3} just gives the leading order behaviour, while the ϕ^0 equation in (53) means that the truncated expansion (52) constitutes an auto-Bäcklund transformation for the nonlinear Schrödinger equation, since $\hat{\psi}$ is another solution of (51). Observe that for x and t real, the singular manifold function ϕ is seen to be real-valued from the leading order behaviour. Since the Painlevé analysis is really concerned with singularities in the space of complex x, t variables, it is more consistent to write the nonlinear Schrödinger equation, together with its complex conjugate, as the system

$$\begin{aligned}
i\psi_t + \psi_{xx} - 2\psi^2\overline{\psi} = 0, \\
-i\overline{\psi}_t + \overline{\psi}_{xx} - 2\overline{\psi}^2\psi = 0,
\end{aligned} \tag{54}$$

and then treat ψ and $\overline{\psi}$ as independent quantities. The system (54) is the first non-trivial flow in the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy [3]. For this full system the singular manifold equations (53) should be augmented with the corresponding 'conjugate' equations: formally these are obtained by taking the complex conjugate with ϕ real (as for real x and t), and α_0, ψ and $\hat{\psi}$ complex. By formally taking the real and imaginary parts of the second equation in (53), which are equivalent to linear combinations of that equation together with its conjugate, the following consequences arise:

$$
\begin{aligned}
\phi_{xx} + \overline{\alpha}_0 \hat{\psi} + \alpha_0 \overline{\hat{\psi}} = 0; \\
i\phi_t + \phi_x (\log[\alpha_0/\overline{\alpha}_0])_x + \overline{\alpha}_0 \hat{\psi} - \alpha_0 \overline{\hat{\psi}} = 0.
\end{aligned}
\tag{55}
$$

Further manipulation of the singular manifold equations (53) and their conjugates, together with (55), leads to the two equations

$$
\alpha_{0,x} = -2i\lambda\alpha_0 - 2\hat{\psi}\phi_x, \tag{56}
$$

$$
i\alpha_{0,t} = (4\lambda^2 + 2\hat{\psi}\overline{\hat{\psi}})\alpha_0 + (-4i\lambda\hat{\psi} + 2\hat{\psi}_x)\phi_x \tag{57}
$$

and their corresponding conjugates, where λ is a constant. Upon substitution of the rearrangement

$$
\alpha_0 = (\psi - \hat{\psi})\phi
$$

of (52) into (56), we find

$$
(\psi - \hat{\psi})_x = -2i\lambda(\psi - \hat{\psi}) - (\psi + \hat{\psi})|\psi - \hat{\psi}|, \tag{58}
$$

where we have used the first equation (53) to substitute $\phi_x = |\alpha_0| = |\psi - \hat{\psi}|$ in the reduction to real x and t. A similar equation for $(\psi - \hat{\psi})_t$ is obtained by eliminating α_0 and ϕ from (57), and the resulting relations between ψ and $\hat{\psi}$ together with (58) constitute a Bäcklund transformation for the nonlinear Schrödinger equation in the form studied by Boiti and Pempinelli, taking the special case $\sigma = 0$ in the formulae of [10]. Starting from the vacuum solution $\hat{\psi} = 0$, and with zero Bäcklund parameter $\lambda = 0$, this BT can be applied repeatedly to obtain a sequence of singular rational solutions of the nonlinear Schrödinger equation, which are described in [48].

The simplest singular rational solution has a single pole, which can be fixed at $x = 0$. If we denote the sequence of these rational solutions $\{\psi_n\}_{n\geq 0}$, then applying the BT (58) with $\lambda = 0$ starting from the vacuum solution the first three are

$$
\psi_0 = 0, \qquad \psi_1 = \frac{1}{x} \qquad \psi_2 = \frac{-2x^3 + 12itx + \tau_3}{x^4 + \tau_3 - 12t^2}, \tag{59}
$$

with τ_3 being an arbitrary constant parameter which is real for real x and t. In general these rational functions can be written as a ratio of polynomial tau-functions $\psi_n = G_n/F_n$ satisfying bilinear equations (see below). The zeros and poles of each ψ_n, which are the roots of the polynomials G_n and F_n respectively, evolve in t according to the equations of Calogero-Moser dynamical systems [48].

As well as leading to the Bäcklund transformation (58) for the nonlinear Schrödinger equation, the singular manifold equations also yield the Lax pair, upon making the squared eigenfunction substitution

$$
\alpha_0 = -\chi_1^2, \qquad \overline{\alpha}_0 = -\chi_2^2. \tag{60}
$$

Fixing a sign we find immediately from the first equation (53) that

$$\phi_x = \chi_1\chi_2,$$

and then putting (60) into (56), (57) and their conjugates gives a matrix system for the vector $\chi = (\chi_1, \chi_2)^T$, that is

$$\begin{aligned} \chi_x &= \mathbf{U}\chi, \\ \chi_t &= \mathbf{V}\chi, \end{aligned} \tag{61}$$

with the matrices

$$\mathbf{U} = \begin{pmatrix} -i\lambda & \psi \\ \overline{\psi} & i\lambda \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} -2i\lambda^2 - i|\psi|^2 & 2\lambda\psi + i\psi_x \\ 2\lambda\overline{\psi} - i\overline{\psi}_x & 2i\lambda^2 + i|\psi|^2 \end{pmatrix}$$

(where we have replaced $\hat{\psi}$ by ψ in $\mathbf{U}$, $\mathbf{V}$). The system (61) is the non-focusing analogue of the Lax pair for the nonlinear Schrödinger equation found by Zakharov and Shabat [104], and for $\mathbf{U}$, $\mathbf{V}$ as above the PDE (51) follows from the compatibility condition for the matrix system, which is the zero curvature equation

$$\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0.$$

For real λ, these matrices are elements of the Lie algebra $su(1,1)$, as opposed to $su(2)$ for the case of the focusing nonlinear Schrödinger equation.

To obtain the Hirota bilinear form of the nonlinear Schrödinger equation we can make a further truncation in (52), setting $\hat{\psi} = 0$, $\alpha_0 = G$, $\phi = F$, so that (51) becomes

$$\frac{1}{F^2}((iD_t + D_x^2)G \cdot F) - \frac{G}{F^3}(D_x^2 F \cdot F + 2|G|^2) = 0.$$

The two equations in brackets can be consistently decoupled to give the bilinear system for the two tau-functions F, G:

$$\begin{aligned} (iD_t + D_x^2)G \cdot F &= 0; \\ D_x^2 F \cdot F + 2|G|^2 &= 0. \end{aligned} \tag{62}$$

It is easy to check that the numerators and denominators in the rational functions (59) are particular solutions of the system (62). The bilinear form of the nonlinear Schrödinger equation was used by Hirota to derive compact expressions for the multi-soliton solutions [45]. A further consequence of (62) is the bilinear equation

$$iD_x D_t F \cdot F - 2D_x G \cdot \overline{G} = i\gamma F^2, \tag{63}$$

with a constant γ. This constant can be removed by a gauge transformation of the tau-functions, rescaling both F and G by $\exp[\gamma xt/2]$. Eliminating G between (63) and (62), the nonlinear Schrödinger equation is then rewritten as a single trilinear equation, expressed as a sum of two determinants, namely

$$\begin{vmatrix} F & F_x & F_t \\ F_x & F_{xx} & F_{xt} \\ F_t & F_{xt} & F_{tt} \end{vmatrix} + \begin{vmatrix} F & F_x & F_{xx} \\ F_x & F_{xx} & F_{3x} \\ F_{xx} & F_{3x} & F_{4x} \end{vmatrix} = 0. \tag{64}$$

The tau-function solution of the trilinear equation (64) is sufficient to determine both the modulus and the argument of the complex amplitude ψ (see [48] and references).

From the preceding results for the KdV and nonlinear Schrödinger equations it should be clear that truncation methods can be extremely powerful in extracting information about integrable PDEs. There are several refinements of the singular manifold method, in particular those involving truncations using Möbius-invariant combinations of ϕ and its derivatives [19, 72], and the use of two singular manifolds for PDEs with two different leading order behaviours [21]. Probably the most elegant and general synthesis of these extended methods is the approach formulated by Pickering [81], who uses expansions in a modified variable satisfying a system of Riccati equations. Truncation methods have even been used to derive Bäcklund transformations for ODEs, in particular Painlevé equations [18]. However, it is uncertain whether such methods can really be made sufficiently general in order to constitute an algorithmic procedure for deriving Lax pairs for integrable sytems. In particular, truncation methods are not directly applicable to integrable PDEs which exhibit movable algebraic branching in their solutions, which are the subject of the next section.

6 Weak Painlevé tests

There are numerous examples of integrable systems which do not have the strong Painlevé property, but which satisfy the weaker criterion that their general solution has at worst movable algebraic branching. Perhaps the simplest example is to consider a Hamiltonian system with one degree of freedom defined by the Hamiltonian (total energy)

$$H = \frac{1}{2}p^2 + V(q),$$

where the potential energy V is a polyomial in q of degree $d \geq 5$. The equations of motion (Hamilton's equations) are

$$\frac{dq}{dt} = p, \qquad \frac{dp}{dt} = -V'(q),$$

which are trivially integrable by a quadrature:

$$t = t_0 + \int^q \frac{dQ}{\sqrt{2(H - V(Q))}}. \tag{65}$$

If the potential energy is normalized so that the leading term of the polynomial is $-2q^d/(d-2)^2$, then with $q(t)$ having a singularity at $t = t_0$ the integral in (65) gives

$$t - t_0 \sim \pm \int^q \frac{(2-d)\, dQ}{2Q^{d/2}} = \pm q^{1-d/2}, \qquad \text{as} \quad q \to \infty.$$

(for a suitable choice of branch in the square root). Thus at leading order we have

$$q \sim \pm (t - t_0)^{2/(2-d)}. \tag{66}$$

For both $d = 2g+1$ (odd) and $d = 2g+2$ (even) q is determined by the hyperelliptic integral (65) corresponding to an algebraic curve of genus g. When $g = 1$ the solution is given in terms of Weierstrass or Jacobi elliptic functions, and both q and p are meromorphic functions of t. However, for a potential of degree 5 or more we have $g \geq 2$, and it is clear from (66) that q has an algebraic branch point at $t = t_0$, since in that case $2/(2-d)$ is a non-integer, negative rational number. In fact it is easy to verify that (66) is the leading order term of an expansion in powers of $(t - t_0)^{2/(d-2)}$. Rather than being meromorphic as in the elliptic case, for $d \geq 5$ the function $q(t)$ is generically single-valued only on a covering of the complex t-plane with an *infinite* number of sheets, and has an infinite number of algebraic branch points (see [2]).

Clearly for potentials of degree 5 or more, this simple Hamiltonian system fails the basic Painlevé test, and yet it is certainly integrable according to any reasonable definition. (Indeed, any Hamiltonian system with one degree of freedom is integrable in the sense that Liouville's theorem holds.) In order to avoid excluding such basic integrable systems from singularity classification, Ramani et al. [86] proposed an extension of the Painlevé property.

Definition 6.1. The weak Painlevé property: *An ODE has the weak Painlevé property if all movable singularities of the general solution have only a finite number of branches.*

There are many examples of finite-dimensional many-body Hamiltonian systems which are Liouville integrable and yet have algebraic branching in their solutions [1, 2]. Among these examples [2] is the geodesic flow on an ellipsoid, which was solved classically by Jacobi [58]. Many other examples, such as those considered by Abenda and Fedorov in [1], arise naturally as stationary or travelling wave reductions of PDEs derived from Lax pairs, in particular those obtained from energy-dependent Schrödinger operators [50]. Thus the corresponding Lax-integrable PDEs have algebraic branching in their solutions, and fail the Weiss-Tabor-Carnevale test described in section 4. It is natural to extend the notion of the weak Painlevé property to PDEs as well, and perform Painlevé analysis on ODEs and PDEs with this property by allowing algebraic branching and rational (not necessarily integer) values for

the resonances. We illustrate this procedure with the example of the Camassa-Holm equation and a related family of PDEs [51] which have peaked solitons (peakons).

The Camassa-Holm equation was derived in [12] by asymptotic methods as an approximation to Euler's equation for shallow water waves, and was shown to be an integrable equation with an associated Lax pair. In the special case when the linear dispersion terms are removed the equation takes the form

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{67}$$

and in this dispersionless limit it admits a weak solution known as a peakon, which has the form

$$u(x,t) = ce^{-|x-ct|}. \tag{68}$$

Note that the notion of a 'weak solution' (as defined in [37], for instance) is completely unrelated to the 'weak' Painlevé property. The peakon solution has a discontinuous derivative at the position of the peak, and the dispersionless Camassa-Holm equation (67) has exact solutions given by a superposition of an arbitrary number of peakons which interact and scatter elastically, just as for ordinary solitons. A detailed analysis of weak solutions of (67) has been performed by Li and Olver [68].

However, the Camassa-Holm equation is an example of an integrable equation which does not satisfy the requirements of Definition 4.1, but instead passes the *weak* Painlevé test. In the neighbourhood of an arbitrary non-characteristic hypersurface $\phi(x,t) = 0$ where the derivatives of u blow up, it admits an expansion with algebraic branching:

$$u(x,t) = -\phi_t/\phi_x + \sum_{j=0}^{\infty} \alpha_j(x,t)\, \phi^{2/3+j/3}. \tag{69}$$

If we regard the branching part $\phi^{2/3}$ as the leading term (since it produces the singularity in the derivatives u_x, u_t on $\phi = 0$), then the resonances are $r = -1, 0, 2/3$ which correspond to the functions ϕ, α_0, α_2 being arbitrary. The Camassa-Holm equation thus satisfies the weak extension of the Weiss-Tabor-Carnevale test, since the expansion (69) is consistent, with the resonance conditions at $r = 0$ and $r = 2/3$ being satisfied. Of course the test is only local, whereas the weak Painlevé property is a global phenomenon, and to prove it rigorously for this PDE would require considerable further analysis. The weak extension of the Painlevé test is still a useful tool, in the sense that if an equation has irrational or complex branching (either at leading order or in its resonances), or if a failed resonance condition introduces logarithmic branching into the general solution, then this is a good indication of non-integrability. Nevertheless, even for ODEs the weak Painlevé property should be applied cautiously as an integrability criterion. For an excellent discussion see [87].

We would now like to apply the weak Painlevé test to a one-parameter family of PDEs that includes (67), before showing the effect that changes of variables can have on singularity structure. We shall consider the family of PDEs

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \tag{70}$$

where the parameter b is constant. These are non-evolutionary PDEs: due to the presence of the u_{xxt} term, (70) is not an evolution equation for u. The (dispersionless) Camassa-Holm equation is the particular member of this family corresponding to $b = 2$. The original reason for interest in this family is that Degasperis and Procesi applied the method of asymptotic integrability [24] and isolated a new equation as satisfying the necessary conditions for integrability up to some order in a multiple-scales expansion. After removing the dispersion terms by combining a Galilean transformation with a shift in u and rescaling, the Degasperis-Procesi equation can be written as

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{71}$$

which is the $b = 3$ case of (70), and it was proved in [25] by construction of the Lax pair that this new equation is integrable. A powerful perturbative extension of the symmetry approach was also applied to the non-evolutionary PDEs (70) in [70], and it was confirmed that only the special cases $b = 2$ (Camassa-Holm) and $b = 3$ (Degasperis-Procesi) fulfill the necessary conditions to be integrable. Hamiltonian structures and the Wahlquist-Estabrook prolongation algebra method for these PDEs have also been treated in detail [51]. Subsequently it has been shown that (after including dispersion) every member of the family (70) arises as a shallow water wave equation [28], except for the special case $b = -1$.

For Painlevé analysis it is convenient to rewrite (70) in the form

$$m_t + um_x + b\,u_x m = 0, \qquad m = u - u_{xx}. \tag{72}$$

To apply the weak Painlevé test, we look for algebraic branching similar to the leading order in (69), with the derivatives of u blowing up on a singular manifold $\phi(x,t) = 0$. Thus we seek the following leading behaviour:

$$u \sim u_0 + \alpha\phi^{\mu}, \qquad \mu \in \mathbb{Z}, \qquad 0 < \mu < 1. \tag{73}$$

Then for the derivatives of u and m as defined in (72) the most singular terms are as follows:

$$u_x \sim \alpha\phi_x\mu\phi^{\mu-1}, \qquad m \sim -\alpha\phi_x^2\mu(\mu-1)\phi^{\mu-2},$$

$$m_x \sim -\alpha\phi_x^3\mu(\mu-1)(\mu-2)\phi^{\mu-3}, \qquad m_t \sim -\alpha\phi_x^2\phi_t\mu(\mu-1)(\mu-2)\phi^{\mu-3}.$$

Substituting these leading orders into (72) we find a balance at order $\phi^{\mu-3}$ between the m_t and um_x terms provided that

$$u_0 = -\phi_t/\phi_x.$$

The next most singular term in the PDE is then at order $\phi^{2\mu-3}$, corresponding to a balance between the um_x and $u_x m$ terms in (72), with coefficient

$$-\alpha^2\phi_x^3\mu(\mu-1)(\mu-2+b\mu),$$

and this is required to vanish giving

$$\mu = \frac{2}{1+b}. \tag{74}$$

Thus we see that for a weak Painlevé expansion with the leading exponent μ being a rational number between zero and one, the most singular terms require that the parameter b should also be rational with

$$b = \frac{2}{\mu} - 1 > 1.$$

To find and test the resonances in an expansion with this leading order, it is sufficient to take the reduced ansatz (42) for ϕ, and then make a perturbation of the leading order terms with parameter ϵ:

$$u \sim \dot{f}(t) + \alpha(t)\phi^{\mu}(1+\epsilon\phi^r), \qquad \phi = x - f(t). \tag{75}$$

Substituting the perturbed expression into (72) and keeping only terms linear in ϵ, we see that terms possibly appearing at order $\phi^{\mu+r-3}$ cancel out automatically (due to the form of u_0), leaving the resonance equation coming from the coefficient of $\phi^{2\mu+r-3}$, which is

$$-\epsilon\alpha^2(r^3 + (2\mu-1)r^2 + 2(\mu-1)r) = 0.$$

Hence the resonances are

$$r = -1,\ 0,\ 2(1-\mu),$$

with μ given in terms of the parameter b by (74).

Having applied the first part of the weak Painlevé test and found a dominant balance and the corresponding values for the resonances, it becomes apparent that the test is completely ineffective as a means to isolate the two integrable cases $b = 2$ and $b = 3$ of (72). Although the leading order resonance $r = 0$ (corresponding to α being arbitrary) is automatically satisfied, the second resonance condition at $r = 2(1-\mu)$ must be checked for every rational value of μ with $0 < \mu < 1$ (or equivalently every rational value of the parameter $b > 1$). If we write μ in its lowest terms as a ratio of positive integers, $\mu = N_1/N_2$, then (73) is the leading part of an expansion for u in all powers of ϕ^{1/N_2}, and as the difference $N_2 - N_1$ increases there is an increasingly large number of terms to compute before the final resonance is reached.

Checking this resonance for the whole countable infinity of rational numbers $b > 1$ seems to be a totally intractable task. Gilson and Pickering showed that all the PDEs within a class including (72) failed every one of a combination of *strong* Painlevé tests [34]. Nevertheless, it is simple to verify that the weak Painlevé test is satisfied for the two particular cases $b = 2, 3$ which are known to be integrable.

However, after a judicious change of variables, involving a transformation of hodograph type, it is still possible to use Painlevé analysis to isolate the two integrable peakon equations. Such transformations have been applied to integrable PDEs with algebraic branching (see [15]) in order to obtain equivalent systems with the strong Painlevé property. That this should be possible is in accordance with the Ablowitz-Ramani-Segur conjecture, but the difficulty lies in finding the correct change of variables. In fact, for a general class of systems that display weak Painlevé behaviour (related to energy-dependent Schrödinger operators) we presented a particular transformation in [50] and, from an examination of a principal balance, we asserted (without proof) that this transformation produced equivalent systems with the strong Painlevé property. However, from a more careful calculation of other balances we have recently observed that this earlier assertion was incorrect [53]. In the case of the Camassa-Holm equation (67), a link to the first negative flow in the KdV hierarchy was found by Fuchssteiner [32], and in [51] it was shown that the appropriate transformation can be extended to (almost) every member of the family of non-evolutionary PDEs (72).

The key to a suitable change of variables for (72) is the fact that for any $b \neq 0$, $\int m^{1/b}\, dx$ is a conserved quantity, with the conservation law

$$p_t = -(pu)_x, \qquad m = -p^b. \tag{76}$$

This allows a reciprocal transformation, defining new independent variables X, T via

$$dX = p\, dx - pu\, dt, \qquad dT = dt. \tag{77}$$

Observe that the closure condition $d^2X = 0$ for the exact one-form dX is precisely (76), and transforming the derivatives yields the new conservation law

$$(p^{-1})_T = u_X. \tag{78}$$

In the old variables, p is related to u by

$$p^b = (\partial_x^2 - 1)u, \tag{79}$$

Replacing ∂_x by $p\partial_X$ and using (78), this means that (79) can be solved for u to give the identity

$$u = -p(\log p)_{XT} - p^b. \tag{80}$$

Finally the conservation law (78) can be written as an equation for p alone, by substituting back for u as in (80) to obtain

$$\frac{\partial}{\partial T}\left(\frac{1}{p}\right)+\frac{\partial}{\partial X}\Big(p(\log p)_{XT}+p^b\Big)=0. \tag{81}$$

Thus we have seen that for each $b \neq 0$, the equation (72) is reciprocally transformed to (81), with the new dependent variable p and new independent variables X, T as in (77). (For more background on reciprocal transformations, see [64].) By making the subsitution $p = \exp(i\eta)$, (81) becomes a generalized equation of sine-Gordon type [51]. The point of making the reciprocal transformation is that we may now apply the *strong* Weiss-Tabor-Carnevale Painlevé test to the equation in these new variables. At leading order near a hypersurface $\phi(X, T) = 0$ there are two types of singularity that can occur in the equation (81), corresponding to p either vanishing or blowing up there:

- $p \sim \alpha\phi$, for $b \geq -1$, with $\alpha = \pm\phi_X^{-1}$ for $b \neq -1$;
- $p \sim \beta\phi^{\mu}$, for $\mu = 2/(1-b) < 1$.

In the first balance, the resonances are $r = -1, 1, 2$. However, if we require the strong Painlevé test to hold we see that we must have $b \in \mathbb{Z}$, since otherwise the p^b term will introduce branching into the expansion in powers of ϕ. The second balance can only hold for $|b| > 1$, but if $b < -1$ then $\mu \notin \mathbb{Z}$, while if $b > 1$ then requiring $\mu = 1 - M$ to be a (negative) integer gives

$$b = \frac{M+1}{M-1}, \qquad M = 2, 3, 4, \ldots \tag{82}$$

From the first balance we require b to be an integer, and the only integer values in the sequence (82) are $b = 2, 3$ (corresponding to $M = 3, 2$ respectively). Interestingly, when the Wahlquist-Estabrook method is applied to (72), this same sequence crops up from purely algebraic considerations [51].

The above analysis shows that the two integrable cases $b = 2, 3$ are isolated immediately just by looking at the leading order behaviour. It is then straightforward to show that for both types of singularity in the equation (81), these two cases fulfill the resonance conditions and thus satisfy the strong Painlevé test. However, the observant reader will notice that further analysis is required to exclude the two special integer values $b = \pm 1$, for which only the first type of singularity arises; this is left as a challenge to the reader.

7 Outlook

It should be apparent from our dicussion that the various Painlevé tests are excellent heuristic tools for identifying whether a given system of differential equations is likely to be integrable or not. However, the strong Painlevé property is clearly too stringent a requirement, since it is not satisfied by a large class of integrable systems which have movable algebraic branch points in their solutions. On the other hand, checking all possible resonances in the

weak Painlevé test can be impractical as a means to isolate integrable systems, and if there are negative resonances then more detailed analysis may be necessary to pick up logarithmic branching [82]. In this short review we have concentrated on methods for detecting movable poles and branch points. However, for equations like (67), the existence of the peakon solution (68) has led to the promising suggestion that Dirichlet series (sums of exponentials) may be a useful means of testing PDEs [84]. Also, although we have only considered singularities of ODEs in the finite complex plane, there are extensive techniques for analysing asymptotic behaviour at infinity [90, 96, 99].

Before closing, we should like to give a brief mention to the fruitful connection between the singularity structure and integrability of discrete systems, in the context of birational maps or difference equations. In the last twenty years, there has been increased interest in discrete integrable systems. Liouville's theorem on integrable Hamiltonian systems extends naturally to the setting of symplectic maps or more generally to Poisson maps or correspondences [11, 97], and many new examples of integrable maps have been found [94]. Grammaticos, Ramani and Papageorgiou introduced a notion of singularity confinement for maps or difference equations [38], which they used very successfully as a criterion to identify discrete analogues of the Painlevé equations, and they proposed that it should be regarded as a discrete version of the Painlevé property.

In order to illustrate singularity confinement, we shall consider the second order discrete equation

$$u_{n+1}(u_n)^2u_{n-1} = \alpha q^n u_n + \beta, \tag{83}$$

which is a non-autonomous version of an equation of the Quispel-Roberts-Thompson type [85], and can be explicitly solved in elliptic functions in the autonomous case $q = 1$ [52]. For $q \neq 1$ the equation (83) can be regarded as a discrete analogue of the first Painleve equation, because if we set $u_n = h^{-2} - y(nh)$, $\alpha = 4h^{-6}$, $\beta = -3h^{-8}$, $q = 1 - h^5/4$ and take the continuum limit $h \to 0$, with $z = nh$ held fixed, then equation (6) arises at leading order in h.

The idea of singularity confinement is that if a singularity is reached upon iteration of a discrete equation or map, then it is possible to analytically continue through it. (This is by analogy with the fact that the solution of an ODE with the Painlevé property has a unique analytic continuation around a movable pole.) In the case of (83), a singularity will be reached if one of the iterates, say u_N, is zero, because this means that the next iterate u_{N+1} is not defined. By redefining α and shifting the index n if necessary, we can take $N = 1$ without loss of generality, so $u_1 = 0$. The vanishing of u_1 requires that at the previous stage $\alpha u_0 + \beta = 0$ must hold. Setting $u_{-1} = a$ (arbitrary) and

$$\alpha u_0 + \beta = \epsilon$$

gives $u_1 \sim \alpha^2\beta^{-2}a^{-1}\epsilon \to 0$ as $\epsilon \to 0$, and the singularity appears at

$$u_2 \sim -\beta^4 a^2 \alpha^{-3} \epsilon^{-2}.$$

However, subsequently we have $u_3 \sim -q^2\alpha^2\beta^{-2}a^{-1}\epsilon$, $u_4 = O(1)$ and further iterates are regular in the limit $\epsilon \to 0$. In this sense, we say that the singularity is confined.

Although the singularity confinement criterion led to the discovery of many new discrete integrable systems (see [88] and references), it was shown by Hietarinta and Viallet that it is not a sufficient condition for integrability [42]. In fact, they found numerous examples of maps of the plane defined by difference equations of the form

$$u_{n+1} + u_{n-1} = f(u_n),$$

for certain rational functions f, which have confined singularities and yet whose orbit structure displays the characteristics of chaos. Other examples of singularity confinement in non-integrable maps can be found in [54]. Nevertheless, it seems that singularity confinement should be a necessary condition for integrability of a suitably restricted class of maps. In fact, Lafortune and Goriely have shown that for birational maps in d dimensions, singularity confinement is a necessary condition for the existence of $d-1$ independent first integrals [67]. Ablowitz, Halburd and Herbst have made an alternative proposal for extending the Painlevé property to difference equations, by using Nevanlinna theory [7, 41], and this has deep connections with various algebraic or arithmetic measures of complexity in discrete dynamics (see [40, 42, 89, 92] and references).

For the reader who is interested in pursuing the subject of Painlevé analysis and its applications to both integrable and non-integrable equations, a number of excellent review articles are to be recommended [29, 66, 73, 87, 95], as well as the proceedings volume [22].

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